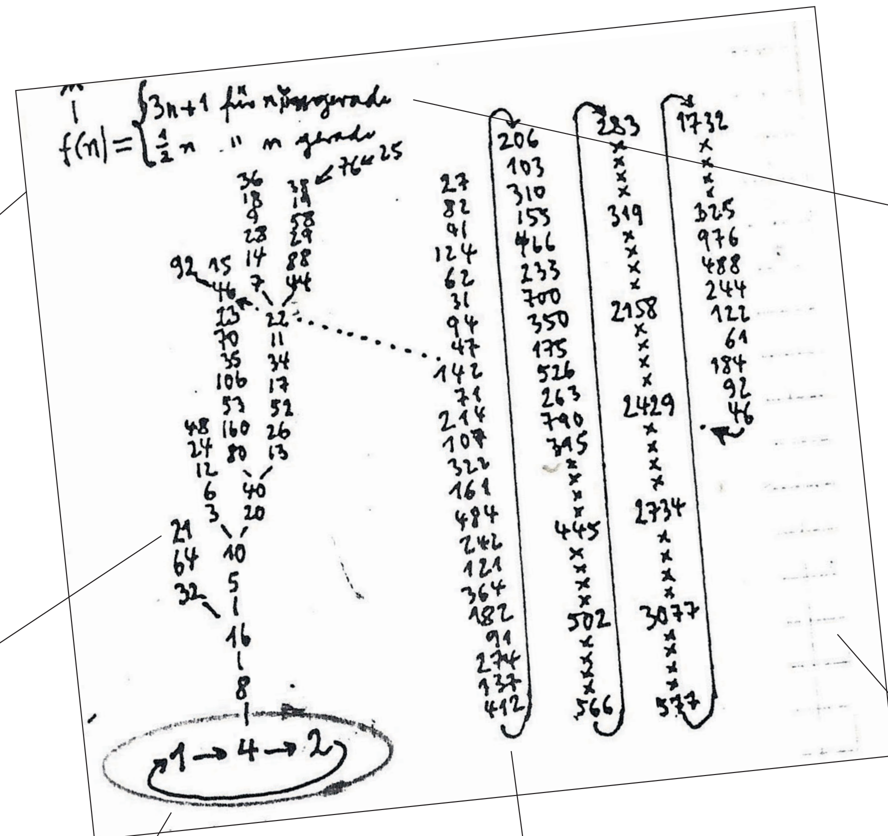
THE DANGEROUS PROBLEM

The century-long struggle to prove the Collatz conjecture

By Joe Kloc

The German mathematician Lothar Collatz doomed many a number theorist when, in 1937, he made his now infamous conjecture: Select any positive integer n. If n is even, divide it by 2; if n is odd, multiply it by 3 and then add 1. Do the same to the result. Do it again. And again. And again. No matter the starting number, Collatz posited, the operation, repeated enough times, will eventually produce the sequence: 4, 2, 1, 4, 2, 1, 4, 2, 1 . . . ad infinitum. This sketch, which Collatz drew in 1980 for the mathematician Michael Mays, demonstrates that his conjecture holds for integers 1 through 28. But proving it true for *all* positive integers, no matter how large, eluded Collatz up until his death in 1990, as it has every attempter since. Known as the "Dangerous Problem," the conjecture has a knack for drawing even the best mathematicians into periods of fruitless work. The Fields Medal winner Terence Tao dedicates a day or two per year to working on it. Mays, who is now retired from the math department of West Virginia University, puts in a few months of effort every half decade or so. "That problem," he said, "has been the source of pleasant reflection and dreams of glory for me for more than fifty years."



The left side of Collatz's sketch demonstrates that, when n is any one of the first twenty-six numbers, it takes at most a couple dozen steps to arrive at the 4–2–1 loop. Let n be 21, for example, and its path is as follows: $f(21) = 64 \rightarrow f(64) = 32 \rightarrow f(32) = 16 \rightarrow f(16) = 8 \rightarrow f(8) = 4 \rightarrow f(4) = 2 \rightarrow f(2) = 1 \rightarrow f(1) = 4 \rightarrow f(4) = 2 \rightarrow f(2) = 1$ . . . and so on. Tao believes that a path to the loop exists for every number, no matter how large. Mays disagrees. What would it mean for either to be proved right? As far as anyone can tell, nothing. Unlike other notorious, unsolved math problems—whose solutions could transform everything from online banking to quantum physics—the Collatz conjecture has no obvious computational or scientific implications. The only reward for proving or disproving it is glory, and a 120 million-yen-prize offered by a Japanese artificial-intelligence firm.

To understand the conjecture's allure, consider the sequence generated when $n = 27$. Unlike the first twenty-six numbers, $f(27)$ requires more than a hundred steps to reach the 4–2–1 loop. As n approaches infinity, this erratic behavior persists—some numbers could theoretically require millions of steps. Supercomputers, through brute-force computation, have managed to show that the conjecture holds for 1 through about 295 quintillion, but one can hardly generalize from such a tiny sliver of the natural number line. In the world of the infinite, no matter how many examples are tested, infinitely more will always remain unchecked. Those who believe that Collatz is wrong often point to a similar example: the Pólya conjecture. Posited by the Hungarian-American mathematician George Pólya in 1919, this conjecture, which deals with prime factorization, was generally assumed to be true. Half a century later, the first counterexample proving it false was identified somewhere near $1{:}85 \times 10^{361}$—almost a googol-cubed times the number of atoms in the universe.

Computing $f(n)$ for small values requires only a pencil and a third grader's grasp of arithmetic, so Collatz wasn't the only one to puzzle over it or to notice its strange behavior. Jeffrey Lagarias, a mathematician at the University of Michigan, found evidence that the conjecture had circulated in math departments during the Cold War. The topologist Shizuo Kakutani relayed that it consumed Yale's mathematicians for an entire month in the early Sixties. Mathematicians at the University of Chicago joked that the problem had been devised by the Soviet Union to slow down mathematical progress in the United States. Lagarias later learned from the Polish physicist Stanisław Ulam that, around the same time, the conjecture had made it out to Los Alamos National Laboratory, where it became a fixation among nuclear researchers. A half century on, little has changed. When, a decade ago, the mathematician Alex Kontorovich told Lagarias that he was interested in proving the Collatz conjecture, Lagarias responded: "Don't fall into that trap." Kontorovich ignored him and continues to wrestle with the problem today. In 2019, Tao announced that he'd gotten "as close as one can get" to a proof before realizing that his method was hopeless. The most recent promising attempt was carried out by a team of computer scientists from Carnegie Mellon and the University of Texas at Austin that approached the problem with automated reasoning. In 2021, they published their results. The abstract read: "We [did] not succeed in proving the Collatz conjecture."

In the early Eighties, Paul Erdős, whose 1,500 published papers make him the most prolific mathematician of the twentieth century, warned that mathematics was "not ready" for the Collatz conjecture. Proving it, Erdős explained, might require a branch of study that doesn't yet exist. His caution was informed by precedent. In 1637, the French mathematician Pierre de Fermat scribbled in the margins of his copy of Diophantus' *Arithmetica* that, for any integer x that is greater than 2, there do not exist any three nonzero numbers a, b, and c such that $a^x + b^x = c^x$. "I have discovered a truly marvelous proof of this," Fermat wrote beside the equation, "which this margin is too narrow to contain." Such a proof has never been found among his papers. If he did have one, it was almost certainly wrong. Finally proving it would take mathematicians another 358 years, more than one hundred peer-reviewed pages, and the formulation of an elliptic-curve theorem not proved until 1986. To some Collatz-addled minds, this is a story of hope: in their quest to prove Fermat's theorem, researchers ended up developing the field of algebraic number theory.

For those who plan to wander out to the far reaches of the infinite number line and prove—or disprove—the Collatz conjecture, the work of the set theorist Georg Cantor offers a cautionary tale: In the 1870s, Cantor set out to prove his continuum hypothesis, which asserted that there was no infinite set that was larger in size than the infinite set of natural numbers but smaller than the infinite set of real numbers. After years of toil, he fell into a depression and devoted his final years to advancing his theory that the British philosopher Francis Bacon was the true author of Shakespeare's plays. After his death, Cantor's continuum hypothesis was taken up by the logician Kurt Gödel, who, in 1940, could establish only that it was mathematically impossible to prove Cantor wrong in standard set theory. Twenty-three years later, a mathematician named Paul Cohen showed that it was also impossible to prove Cantor right. The hypothesis had always been unprovable, and it always will be. ■

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